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1-absorbing primary submodules

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Abstract

Let R be a commutative ring with non-zero identity and M be a unitary R-module. The goal of this paper is to extend the concept of 1-absorbing primary ideals to 1-absorbing primary submodules. A proper submodule N of M is said to be a 1-absorbing primary submodule if whenever non-unit elements $a, b \in R$ and $m \in M$ with $abm \in N$, then either $ab \in (N :_R M)$ or $m \in M - rad(N)$. Various properties and chacterizations of this class of submodules are considered. Moreover, 1-absorbing primary avoidance theorem is proved.

1 Introduction

Throughout this paper, we shall assume unless otherwise stated, that all rings are commutative with non-zero identity and all modules are considered to be unitary. A *prime* (resp. *primary*) submodule is a proper submodule N of M with the property that for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M)$ (resp. $a \in \sqrt{(N :_R M)}$). Since prime and primary ideals (submodules) have an important role in the theory of modules over commutative rings, generalizations of these concepts have been studied by several authors [1]-[8], [14], [15]. For a survey article consisting some of generalizations see [5]. In 2007, Badawi [4] called a non-zero proper ideal I of R a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. As an extension of 2-absorbing primary ideals, the concept of 2-absorbing submodules are introduced by Darani and Soheilnia [8] and studied by Payrovi,

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Babaei [15]. We recall from [8] that a proper submodule N of M is said to be a *2-absorbing submodule* if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. In 2014, Badawi, Tekir and Yetkin [6] introduced the concept of 2-absorbing primary ideals. A proper ideal I of R is called *2-absorbing primary* if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. After that, the notion of 2-absorbing pimary submodules is introduced and studied in [14]. According to [14], a proper submodule N of M is said to be *2-absorbing primary* provided that $a, b \in R$, $m \in M$ and $abm \in N$ imply either $ab \in (N :_R M)$ or $am \in M\text{-}rad(N)$ or $bm \in M\text{-}rad(N)$. As a recent study, the class of *1-absorbing primary ideals* was defined in [7]. According to [7], a proper ideal I of R is said to be a *1-absorbing primary ideal* if whenever non-unit elements a, b, c of R and $abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$. Our aim is to extend the notion of 1-absorbing primary ideals to 1-absorbing primary submodules.

For the sake of thoroughness, we give some definitions which we will need throughout this study. Let I be an ideal of a ring R. By $\sqrt{I}$, we mean the radical of I which is the intersection of all prime ideals containing I, that is $\{r \in R : r^n \in I \text{ for some } n\}$. Let M be an R-module and N be a submodule of M. We will denote by $(N :_R M)$ *the residual of* N *by* M, that is, the set of all $r \in R$ such that $rM \subseteq N$. The annihilator of M denoted by $Ann_R(M)$ is $(0 :_R M)$. The M-radical of N, denoted by M-rad(N), is defined to be the intersection of all prime submodules of M containing N. If M is a multiplication R-module, then $M - rad(N) = \{m \in M : m^k \subseteq N$ for some $k \geq 0\}$ [1, Theorem 3.13]. If there is no such a prime submodule, then $M - rad(N) = M$. For the other notations and terminologies that are used in this article, the reader is referred to [10].

We summarize the content of this article as follows. We call a proper submodule N of M a 1-absorbing primary submodule if whenever non-unit elements $a, b \in R$ and $m \in M$ with $abm \in N$, then $ab \in (N :_R M)$ or $m \in M - rad(N)$. It is clear that a prime submodule is a 1-absorbing primary submodule, and a 1-absorbing primary submodule is a 2-absorbing primary submodule. In Section 2, we start with examples (Example 1 and Example 2) showing that the inverses of these implications are not true in general. Various characterizations for 1-absorbing primary submodules are given (Theorem 1, Theorem 2 and Theorem 3). Moreover, the behavior of 1-absorbing primary submodules in modules under homomorphism, module localizations and direct product of modules are investigated (Proposition 3, Proposition 4 and Proposition 5). Finally, in Section 3, the 1-absorbing primary avoidance theorem is proved.

2 Properties of 1-absorbing primary submodules

Definition 1. *Let M be a module over a commutative ring R and N be a proper submodule of M. We call N a 1-absorbing primary submodule if whenever non-unit elements $a, b \in R$ and $m \in M$ with $abm \in N$, then $ab \in (N :_R M)$ or $m \in M - rad(N)$.*

It is clear that the following implication hold: prime submodule $\Rightarrow$ 1-absorbing primary submodule $\Rightarrow$ 2-absorbing primary submodule. The following example shows that a 1-absorbing primary submodule of M needs not to be a primary (prime) submodule; and also there are 2-absorbing primary submodules which are not 1-absorbing primary.

Example 1. 1. *Let $A = K[x, y]$, where K is a field, $Q = (x, y)A$. Consider $R = A_Q$ and $M = R$ as an R-module. Then $N = (x^2, xy)M$ is a 1-absorbing primary submodule of M [7, Example 1]. Observe that $\sqrt{(N :_R M)} = xR$. Since $x \cdot y \in N$, but $x \notin N$ and $y \notin \sqrt{(N :_R M)}$, N is not a primary submodule (so, it is not a prime submodule) of M.*

2. *Consider the submodule $N = p^2 q\mathbb{Z}$ of $\mathbb{Z}$-module $\mathbb{Z}$ where p and q are distinct prime integers. Then N is a 2-absorbing primary submodule of $\mathbb{Z}$ by [14, Corollary 2.21]. However it is not 1-absorbing primary as $p \cdot p \cdot q \in N$ but neither $p \cdot p \in (N :_{\mathbb{Z}} \mathbb{Z}) = N$ nor $q \in M - rad(N) = pq\mathbb{Z}$.*

The following example shows that there are some modules of which every proper submodule is 2-absorbing primary but which has no 1-absorbing primary submodule.

Example 2. *Let p be a fixed prime integer. Then $E(p) = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = r/p^n + \mathbb{Z}$ for some $r \in \mathbb{Z}$ and $n \in \mathbb{N} \cup (0)\}$ is a non-zero submodule of $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. For each $t \in \mathbb{N}$, set $G_t = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = r/p^t + Z$ for some $r \in \mathbb{Z}\}$. Observe that each proper submodule of $E(p)$ is equal to G_i for some $i \in \mathbb{N}$ and $(G_t :_{\mathbb{Z}} E(p)) = 0$ for every $t \in \mathbb{N}$. It is shown in [3, Example 1] that every submodule G_t is not a primary submodule of $E(p)$. Thus there is no prime submodule in $E(p)$. Thus $E(p) - rad(G_t) = E(p)$. Therefore, each G_t is a 1-absorbing primary submodule of R.*

We next give several characterizations of 1-absorbing primary submodules of an R-module.

Theorem 1. *Let N be a proper submodule of an R-module M. Then the following statements are equivalent:*

1. N is a 1-absorbing primary submodule of M.

2. If a, b are non-unit elements of R such that $ab \notin (N :_R M)$, then $(N :_M ab) \subseteq M - rad(N)$.

3. If a, b are non-unit elements of R, and K is a submodule of M with $abK \subseteq N$, then $ab \in (N :_R M)$ or $K \subseteq M - rad(N)$.

4. If $I_1 I_2 K \subseteq N$ for some proper ideals I_1, I_2 of R and some submodule K of M, then either $I_1 I_2 \subseteq (N :_R M)$ or $K \subseteq M - rad(N)$.

Proof. (1)⇒(2) Suppose that a, b are non-unit elements of R such that $ab \notin (N :_R M)$. Let $m \in (N :_M ab)$. Hence $abm \in N$. Since N is 1-absorbing primary submodule and $ab \notin (N :_R M)$, we have $m \in M - rad(N)$, and so $(N :_M ab) \subseteq M - rad(N)$.

(2)⇒(3) Suppose that $ab \notin (N :_R M)$. Since $abK \subseteq N$, we have $K \subseteq (N :_M ab) \subseteq M - rad(N)$ by (2).

(3)⇒(4) Assume on the contrary that neither $I_1 I_2 \subseteq (N :_R M)$ nor $K \subseteq M - rad(N)$. Then there exist non-unit elements $a \in I_1$, $b \in I_2$ with $ab \notin (N :_R M)$. Thus $abK \subseteq N$, it contradicts with (3).

(4)⇒(1) Let $a, b \in R$ be non-unit elements, $m \in M$ and $abm \in N$. Put $I_1 = aR$, $I_2 = bR$, $K = Rm$. Thus the result is clear. □

An R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R. Equivalently, $N = (N :_R M)M$ [9]. Let M be a multiplication R-module and let $N = IM$ and $K = JM$ for some ideals I and J of R. The product of N and K is denoted by NK is defined by IJM. Clearly, NK is a submodule of M and contained in $N \cap K$. It is shown in [1, Theorem 3.4] that the product of N and K is independent of presentations of N and K. It is shown in [9, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then M-rad$(N) = \sqrt{(N :_R M)}M$. If M is a finitely generated multiplication R-module, then $(M\text{-rad}(N) : M) = \sqrt{(N :_R M)}$ by [14, Lemma 2.4]. Now, we are ready for characterizing 1-absorbing primary submodules of finitely generated multiplication module M in terms of submodules of M.

Theorem 2. *Let M be a finitely generated multiplication R-module and N be a proper submodule of M. Then the following statements are equivalent:*

1. N is a 1-absorbing primary submodule of M.

2. If $N_1 N_2 N_3 \subseteq N$ for some submodules N_1, N_2, N_3 of M, then either $N_1 N_2 \subseteq N$ or $N_3 \subseteq M - rad(N)$.

Proof. (1)⇒(2) Suppose that N is a 1-absorbing primary submodule of M, $N_1 N_2 N_3 \subseteq N$ and $N_3 \nsubseteq M - rad(N)$. Since M is a finitely generated multiplication module, $N_1 = I_1 M$ and $N_2 = I_2 M$ for some ideals I_1, I_2 of R. Hence

$I_1I_2N_3 \subseteq N$. Since $N_3 \nsubseteq M-rad(N)$, we have $I_1I_2 \subseteq (N :_R M)$ by Theorem 1. Thus we conclude $N_1N_2 \subseteq (N : M)M = N$.

(2)⇒(1) Let $I_1I_2K \subseteq N$. Then there exists an ideal I_3 of R such that $I_1I_2I_3M \subseteq N$ which gives $I_1I_2M \subseteq N$ or $I_3M \subseteq M - rad(N)$. By [16, p.231 Corollary], we have $I_1I_2 \subseteq (N :_R M) + Ann_R(M) = (N :_R M)$ or $K \subseteq M - rad(N)$. Hence, we are done from Theorem 1. □

Lemma 1. *[16, Theorem 10]Let M be a finitely generated faithful multiplication R-module, then $(IM : M) = I$ .for all ideals I of R.*

In [3, Corollary 2], for a proper submodule N of a multiplication R-module M, it is shown that N is primary submodule of M if and only if $(N :_R M)$ is primary ideal of R. Analogous with this result, we have the following.

Theorem 3. *Let I be an ideal of a ring R and N be a submodule of a finitely generated faithful multiplication R-module M. Then*

1. I is a 1-absorbing primary ideal of R if and only if IM is a 1-absorbing primary submodule of M.

2. N is a 1-absorbing primary submodule of M if and only if $(N : M)$ is a 1-absorbing primary ideal of R.

3. N is a 1-absorbing primary submodule of M if and only if $N = IM$ for some 1-absorbing primary ideal of R.

Proof. (1) Suppose I is a 1-absorbing primary ideal of R. If $IM = M$, then $I = (IM : M) = R$ by Lemma 1, a contradiction. Thus, IM is proper in M. Now, let $a, b \in R$ be non-unit elements and $m \in M$ such that $abm \in IM$ and $ab \notin (IM : M) = I$. Then $ab((m) : M) = ((abm) : M) \subseteq (IM : M) \subseteq (\sqrt{I}M : M) = \sqrt{I}$. Since I is a 1-absorbing primary ideal, we conclude that $((m) : M) \subseteq \sqrt{I}$. Thus, $m \in ((m) : M)M \subseteq \sqrt{I}M = M\text{-}rad(IM)$. Conversely, suppose IM is 1-absorbing primary submodule of M. Then clearly I is proper in R. Let $a, b, c \in R$ be non-unit elements with $abc \in I$ and $ab \notin (IM : M) = I$. Since $abM \in IM$ and IM is a 1-absorbing primary submodule, then $cM \subseteq M\text{-}rad(IM) = \sqrt{I}M$. Therefore, $c \in (\sqrt{I}M : M) = \sqrt{I}$ and I is a 1-absorbing primary ideal of R.

(2) Since $N = (N : M)M$, it follows by (1).

(3) Putting $I = (N : M)$ in (2), the claim is clear. □

The following example shows that if $(N :_R M)$ is a 1-absorbing primary ideal of R, then N is not needed to be a 1-absorbing primary submodule in general.

Example 3. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \times \mathbb{Z}$ be an R-module and p a prime integer. Consider the submodule $N = p^n\mathbb{Z} \times \{0\}$ of M for $n \geq 2$. Then $(N :_R M) = \{0\}$ is a 1-absorbing primary ideal of R. However, N is not a 1-absorbing primary submodule of M since $p \cdot p^{n-1} \cdot (1,0) \in N$ but neither $p \cdot p^{n-1} = p^n \in (N :_R M) = \{0\}$ nor $(1,0) \in M - rad(N) = p\mathbb{Z} \times \{0\}$.*

In view of Theorem 3, we conclude the following result.

Proposition 1. *Let M be a finitely generated multiplication R-module and N be a 1-absorbing primary submodule of M. Then the following are satisfied:*

1. $\sqrt{(N :_R M)}$ is a prime ideal of R.
2. $\sqrt{(N :_R m)}$ is a prime ideal of R containing $\sqrt{(N :_R M)} = P$ for every $m \notin M - rad(N)$.
3. $M - rad(N)$ is a prime submodule of M.

Proof. (1) Let N be a 1-absorbing primary submodule of M. Then $(N :_R M)$ is 1-absorbing primary ideal of R by Theorem 3. From [7, Theorem 2], we conclude that $\sqrt{(N :_R M)}$ is a prime ideal of R.

(2) Since N is a 1-absorbing primary ideal, $\sqrt{(N :_R M)} = P$ is a prime ideal of R by (1). Suppose that $a, b \in R$ such that $ab \in \sqrt{(N :_R m)}$. Without loss of generality we may assume that a and b are non-unit elements of R. Then there exists a positive integer n such that $a^n b^n m \in N$. Since N is 1-absorbing primary submodule, and $m \notin M - rad(N)$, it implies that either $(ab)^n \in (N :_R M)$. Since P is prime and $ab \in P$, we conclude either $a \in P = \sqrt{(N :_R M)} \subseteq \sqrt{(N :_R m)}$ or $b \in P = \sqrt{(N :_R M)} \subseteq \sqrt{(N :_R m)}$.

(3) Suppose thet N is a 1-absorbing primary submodule. Since $\sqrt{(N :_R M)}$ is a prime ideal of R by (1), we conclude that $M - rad(N) = \sqrt{(N :_R M)}M$ is a prime submodule of M by [9, Corollary 2.11]. $\square$

Note that the intersection of two distinct non-zero 1-absorbing primary submodules need not be a 1-absorbing primary submodule. Consider $\mathbb{Z}$-module $\mathbb{Z}$. Then $2\mathbb{Z}$ and $3\mathbb{Z}$ are clearly 1-absorbing primary submodules but $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ is not. Indeed, $2 \cdot 2 \cdot 3 \in 6\mathbb{Z}$ but neither $2 \cdot 2 \in (\mathbb{Z} : 6\mathbb{Z}) = 6\mathbb{Z}$ nor $3 \in \mathbb{Z} - rad(6\mathbb{Z}) = 6\mathbb{Z}$. We call a proper submodule N of M a P-1-absorbing submodule of M if $\sqrt{(N :_R M)} = P$ is a prime submodule of R. In the next theorem, we show that if N_i's are P-1-absorbing primary submodules of a multiplication module M, then the intersection of these submodules is a P-1-absorbing primary submodule of M.

Proposition 2. *Let M be a multiplication R-module. If $\{N_i\}_{i=1}^{k}$ is a family of P-1-absorbing primary submodules of M, then so is $\bigcap_{i=1}^{k} N_i$.*

Proof. Suppose that $abm \in \bigcap_{i=1}^{k} N_i$ but $ab \notin \left(\bigcap_{i=1}^{k} N_i :_R M\right)$ for non-unit elements $a, b \in R$ and $m \in M$. Then $ab \notin (N_j :_R M)$ for some $j \in \{1, ..., k\}$. Since N_j is 1-absorbing primary and $abm \in N_j$, we have $m \in M - rad(N_j)$. Now, since $M - rad\left(\bigcap_{i=1}^{k} N_i\right) = \bigcap_{i=1}^{k} M - rad(N_i) = PM$ by [14, Proposition 2.14 (3)], we are done. □

Lemma 2. *[12]Let $\varphi : M_1 \longrightarrow M_2$ be an R-module epimorphism. Then*

1. *If N is a submodule of M_1 and $\ker(\varphi) \subseteq N$, then $\varphi(M_1\text{-}rad(N)) = M_2\text{-}rad(\varphi(N))$.*
2. *If K is a submodule of M_2, then $\varphi^{-1}(M_2\text{-}rad(K)) = M_1\text{-}rad(\varphi^{-1}(K))$.*

Proposition 3. *Let M_1 and M_2 be R-modules and $f : M_1 \to M_2$ be a module homomorphism. Then the following statements hold:*

1. If N_2 is a 1-absorbing primary submodule of M_2, then $f^{-1}(N_2)$ is a 1-absorbing primary submodule of M_1.
2. Let f be an epimorphism. If N_1 is a 1-absorbing primary submodule of M_1 containing $Ker(f)$, then $f(N_1)$ is a 1-absorbing primary submodule of M_2.

Proof. (1) Suppose that a, b are non-unit elements of R, $m_1 \in M_1$ and $abm_1 \in f^{-1}(N_2)$. Then $abf(m_1) \in N_2$. Since N_2 is 1-absorbing primary, we have either $ab \in (N_2 :_R M_2)$ or $f(m_1) \in M_2 - rad(N_2)$. Here, we show that $(N_2 :_R M_2) \subseteq (f^{-1}(N_2) :_R M_1)$. Let $r \in (N_2 :_R M_2)$. Then $rM_2 \subseteq N_2$ which implies that $rf^{-1}(M_2) \subseteq f^{-1}(N_2)$; i.e. $rM_1 \subseteq f^{-1}(N_2)$. Thus $r \in (f^{-1}(N_2) :_R M_1)$. Hence $ab \in (f^{-1}(N_2) :_R M_1)$ or $m_1 \in f^{-1}(M_2 - rad(N_2))$. Since $f^{-1}(M_2 - rad(N_2)) = M_1 - rad(f^{-1}(N_2))$ by Lemma 2 (2) and $f^{-1}(N_2)$ is a 1-absorbing primary submodule of M_1.

(2) Suppose that a,b are non-unit elements of R, $m_2 \in M_2$ and $abm_2 \in f(N_1)$. Since f is an epimorphism, there exists $m_1 \in M_1$ such that $f(m_1) = m_2$. Since $Kerf \subseteq N_1$, $abm_1 \in N_1$. Hence $ab \in (N_1 :_R M_1)$ or $m_1 \in M_1 - rad(N_1)$. Here, we show that $(N_1 :_R M_1) \subseteq (f(N_1) :_R M_2)$. Let $r \in (N_1 :_R M_1)$. Then $rM_1 \subseteq N_1$ which implies that $rf(M_1) \subseteq f(N_1)$. Since f is

onto, we conclude that $rM_2 \subseteq f(N_1)$, that is, $r \in (f(N_1) :_R M_2)$. Thus $ab \in (f(N_1) :_R M_2)$ or $m_2 = f(m_1) \in f(M_1 - rad(N_1)) = M_2 - rad(f(N_1))$ by Lemma 2 (1), as desired. $\square$

As a consequence of Theorem 3, we have the following result.

Corollary 1. *Let M be an R-module and N_1, N_2 be submodules of M with $N_2 \subseteq N_1$. Then N_1 is a 1-absorbing primary submodule of M if and only if N_1/N_2 is a 1-absorbing primary submodule of M/N_2.*

Proof. Suppose that N_1 is a 1-absorbing primary submodule of M. Consider the canonical epimorphism $f : M \to M/N_2$ in Proposition 3. Then N_1/N_2 is a 1-absorbing primary submodule of M/N_2. Conversely, let a and b are non-unit elements of R, $m \in M$ such that $abm \in N_1$. Hence $ab(m + N_2) \in N_1/N_2$. Since N_1/N_2 is a 1-absorbing primary submodule of M/N_2, it implies either $ab \in (N_1/N_2 :_R M/N_2)$ or $m+N_2 \in M/N_2-rad(N_1/N_2) = M-rad(N_1)/N_2$. Therefore $ab \in (N_1 :_R M)$ or $m \in M - rad(N_1)$. Thus N_1 is a 1-absorbing primary submodule of M. $\square$

Let M_1 be R_1-module and M_2 be R_2-module where R_1 and R_2 are commutative rings with identity. Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then M is an R-module and every submodule of M is of the form $N = N_1 \times N_2$ for some submodules N_1, N_2 of M_1, M_2, respectively. Also, $M - rad(N_1 \times N_2) = M_1 - rad(N_1) \times M_2 - rad(N_2)$ by [2, Lemma 2.3 (ii)].

Proposition 4. *Let M_1 be an R_1-module and M_2 be an R_2-module, where R_1, R_2 are commutative rings with identity, $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Suppose that N_1 is a proper submodule of M_1. If $N = N_1 \times M_2$ is a 1-absorbing primary submodule of R-module M, then N_1 is a 1-absorbing primary submodule of R_1-module M_1.*

Proof. Suppose that $N = N_1 \times M_2$ is a 1-absorbing primary submodule of M. Put $M' = M/\{0\} \times M_2$ and $N' = N/\{0\} \times N_2$. From Corollary 1, N' is a 1-absorbing primary submodule of M'. Since $M' \cong M_1$ and $N' \cong N_1$, we conclude the result. $\square$

Proposition 5. *Let S be a multiplicatively closed subset of a commutative ring R and M be an R-module. If N is a 1-absorbing primary submodule of M and $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a 1-absorbing primary submodule of $S^{-1}R$-module $S^{-1}M$.*

Proof. Let $\frac{a}{s_1}$ and $\frac{b}{s_2}$ be non-unit elements of $S^{-1}R$, $\frac{m}{s_3} \in S^{-1}M$ with $\frac{a}{s_1}\frac{b}{s_2}\frac{m}{s_3} \in S^{-1}N$. Hence $tabm \in N$ for some $t \in S$. Since N is 1-absorbing primary, we have either $tm \in M - rad(N)$ or $ab \in (N :_R M)$. Thus we conclude either

$\frac{m}{s_3} = \frac{tm}{ts_3} \in S^{-1}(M - rad(N)) \subseteq S^{-1}M - rad(S^{-1}N)$ or $\frac{ab}{s_1 s_2} \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$. □

Let R be a ring and M be an R-module. The idealization of M is denoted by $R(M) = R(+)M$ is a commutative ring with identity with coordinate-wise addition and multiplication defined by $(a, m_1)(b, m_2) = (ab,\, am_2 + bm_1)$. An ideal H is called homogeneous if $H = I(+)N$ for some ideal I of R and some submodule N of M such that $IM \subseteq N$.

Proposition 6. *Let M be an R-module and $I(+)N$ be a homogeneous ideal of $R(M)$. If $I(+)N$ is a 1-absorbing primary ideal of $R(M)$, then I is a 1-absorbing primary ideal of R.*

Proof. Suppose that a, b, c are non-unit elements of R such that $abc \in I$ and $c \notin \sqrt{I}$. Then $(a, 0_M) \cdot (b, 0_M) \cdot (c, 0_M) \in I(+)N$. Note that $\sqrt{I(+)N} = \sqrt{I}(+)M$ by [10, Theorem 25.1 (5)]. Then $(c, 0_M) \notin \sqrt{I(+)N}$. Since $I(+)N$ is 1-absorbing primary, we conclude that $(a, 0_M) \cdot (b, 0_M) \in I(+)N$. Thus $ab \in I$, we are done. □

3 The 1-absorbing primary avoidance theorem

In this section, we prove the 1-absorbing primary avoidance theorem. Throughout this section, let M be a finitely generated multiplication R-module and N, $N_1, ..., N_n$ be submodules of M. Recall from [11] that a covering $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ is said to be efficient if no N_k is superfluous. Also, $N = N_1 \cup N_2 \cup \cdots \cup N_n$ is an efficient union if none of the N_k may be excluded. A covering of a submodule by two submodules is never efficient.

Theorem 4. *Let $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ be an efficient covering of submodules $N_1, N_2, ..., N_n$ of M where $n > 2$. If $\sqrt{(N_i :_R M)} \nsubseteq \sqrt{(N_j :_R m)}$ for all $m \in M \backslash M - rad(N_j)$ whenever $i \neq j$, then no N_i ($1 \leq i \leq n$) is a 1-absorbing primary submodule of M.*

Proof. Assume on the contrary that N_k is a 1-absorbing primary submodule of M for some $1 \leq k \leq n$. Since $N \subseteq \cup N_i$ is an efficient covering, $N \subseteq \bigcup (N_i \cap N)$ is also an efficient covering. From [11, Lemma 2.1], $\left(\bigcap_{i \neq k} N_i \right) \cap N \subseteq N_k \cap N$. Here, observe that $\sqrt{(N_i :_R M)}$ is a proper ideal of R for all $1 \leq i \leq n$. Also, from our assumption, there is a non-unit element $a_i \in \sqrt{(N_i :_R M)} \backslash \sqrt{(N_k :_R m)}$ for all $i \neq k$ and for all $m \in M \backslash M - rad(N_k)$. Then there is a positive integer n_i such that $a_i^{n_i} \in (N_i :_R M)$ for each $i \neq k$. Put

$a = \prod_{i=1}^{k-1} a_i$, $b = \prod_{i=k+1}^{n} a_i$ and $n = \max\{n_1, ..., n_{k-1}, n_{k+1}, ..., n_n\}$. Now, we show that $a^n b^n m \in \left(\left(\bigcap_{i \neq k} N_i\right) \cap N\right) \setminus (N_k \cap N)$. Suppose that $a^n b^n m \in N_k \cap N$. Then $a^n b^n \in (N_k :_R m) \subseteq \sqrt{(N_k :_R m)}$. By Theorem 1 (2), $\sqrt{(N_k :_R m)}$ is a prime ideal. It implies that $a \in \sqrt{(N_k :_R m)}$ or $b \in \sqrt{(N_k :_R m)}$. Thus $a_i \in \sqrt{(N_k :_R m)}$ for some $i \neq k$, a contradiction. Therefore $a^n b^n m \in \left(\left(\bigcap_{i \neq k} N_i\right) \cap N\right) \setminus (N_k \cap N)$ which is a contradiction. Thus N_k is not a 1-absorbing primary submodule. □

Theorem 5. *(1-absorbing Primary Avoidance Theorem) Let N, N_1, $N_2, ..., N_n$ $(n \geq 2)$ be submodules of M such that at most two of $N_1, N_2, ..., N_n$ are not 1-absorbing primary with $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$. If $\sqrt{(N_i :_R M)} \nsubseteq \sqrt{(N_j :_R m)}$ for all $m \in M \backslash M - rad(N_j)$ whenever $i \neq j$, then $N \subseteq N_k$ for some $1 \leq k \leq n$.*

Proof. Since it is clear for $n \leq 2$, suppose that $n > 2$. Since any cover consisting submodules of M can be reduced to an efficient one by deleting any unnecessary terms, we may assume that $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ is an efficient covering of submodules of M. From Theorem 4, it implies that no N_k is a 1-absorbing primary submodule which contradicts with the hypothesis. Thus $N \subseteq N_k$ for some $1 \leq k \leq n$. □

Corollary 2. *Let N be a proper submodule of M. If 1-absorbing primary avoidance theorem holds for M, then the 1-absorbing primary avoidance theorem holds for M/N.*

Proof. Let K/N, N_1/N, $N_2/N, ..., N_n/N$ $(n \geq 2)$ be submodules of M/N such that at most two of $N_1/N, N_2/N, ..., N_n/N$ are not 1-absorbing primary and $K/N \subseteq N_1/N \cup N_2/N \cup \cdots \cup N_n/N$. Hence, $K \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ and at most two of N_1, $N_2, ...,$ N_n are not 1-absorbing primary by Corollary 1. Suppose that
$\sqrt{(N_i/N :_R M/N)} \nsubseteq \sqrt{(N_j/N :_R m + N)}$ for all $m + N \in (M/N) \backslash (M - rad(N_j/N))$ whenever $i \neq j$. It is easy to verify that if $\sqrt{(N_i :_R M)} \subseteq \sqrt{(N_j :_R m)}$ for some $m \in M$, then $\sqrt{(N_i/N :_R M/N)} \subseteq \sqrt{(N_j/N :_R m + N)}$ for some $m + N \in M/N$. Also observe that if $m + N \in (M/N) \backslash (M/N - rad(N_j/N)) = (M/N) \backslash (M - rad(N_j)/N)$, then $m \in M \backslash M - rad(N_j)$. Thus, from our assumption $\sqrt{(N_i/N :_R M/N)} \nsubseteq \sqrt{(N_j/N :_R m + N)}$ for all $m + N \in (M/N) \backslash (M/N - rad(N_j/N))$ whenever $i \neq j$, we conclude that

$\sqrt{(N_i :_R M)} \nsubseteq \sqrt{(N_j :_R m)}$ for all $m \in M \backslash M - rad(N_j)$ whenever $i \neq j$. From our hypothesis and Theorem 5, we have $K \subseteq N_k$ for some $1 \leq k \leq n$.. Consequently, $K/N \subseteq N_k/N$ for some $1 \leq k \leq n$; so we are done. □

In view of Theorem 4 and Theorem 5, we conclude 1-absorbing primary avoidance theorem for rings.

Corollary 3. *Let $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ be an efficient covering of ideals $I_1, I_2, ..., I_n$ of a ring R where $n > 2$. If $\sqrt{I_i} \nsubseteq \sqrt{(I_j : x)}$ for all $x \in R \backslash \sqrt{I_j}$ whenever $i \neq j$, then no I_i $(1 \leq i \leq n)$ is a 1-absorbing primary ideal of R.*

Corollary 4. *(1-absorbing Primary Avoidance Theorem for Rings) Let $I, I_1, I_2, ..., I_n$ $(n \geq 2)$ be ideals of a ring R such that at most two of $I_1, I_2, ..., I_n$ are not 1-absorbing primary with $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$. If $\sqrt{I_i} \nsubseteq \sqrt{(I_j : x)}$ for all $x \in R \backslash \sqrt{I_j}$ whenever $i \neq j$, then $I \subseteq I_k$ for some $1 \leq k \leq n$.*

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